

ADDING RESISTANCES AND CAPACITANCES IN INTRODUCTORY ELECTRICITY

C.J. Efthimiou¹ and R.A. Llewellyn²

Department of Physics
University of Central Florida
Orlando, FL 32826

Abstract

We propose a unified approach to addition of resistors and capacitors such that the formulæ are always simply additive. This approach has the advantage of being consistent with the intuition of the students. To demonstrate our point of view, we re-work some well-known end-of-the-chapter textbook problems and propose some additional new problems.

1 Introduction

All introductory physics textbooks, with or without calculus, cover the addition of both resistances and capacitances in series and in parallel. The formulæ for adding resistances

$$R = R_1 + R_2 + \dots, \quad (1)$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \dots, \quad (2)$$

and capacitances

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \dots, \quad (3)$$

$$C = C_1 + C_2 + \dots, \quad (4)$$

are well-known and well-studied in all the books.

In books with calculus there are often end-of-chapter problems in which students must find R and C using continuous versions of equations (1) and (4) [2, 5, 6, 7, 8]. However, we have found *none* which includes problems that make use of continuous versions of equations (2) and (3) [2, 3, 4, 5, 6, 7, 8]. Students who can understand and solve the first class of problems should be able to handle the second class of the problems, as well. We feel that continuous problems that make use of all four equations should be shown to the students in order to give them a global picture of how calculus is applied to physical problems. Physics contains much more than mathematics. When integrating quantities in physics, the way we integrate them

¹costas@physics.ucf.edu

²ral@physics.ucf.edu

is motivated by the underlying physics. Students often forget the physical reasoning and they tend to add (integrate) quantities only in one way.

In this paper, we introduce an approach to solving continuous versions of equations (2) and (3) that is as straightforward and logical for the students as solving continuous versions of equations (1) and (4). We then present some problems in which the student must decide which formula is the right one to use for integration. We hope that this article will motivate teachers to explain to students the subtle points between ‘straight integration’ as taught in calculus and ‘physical integration’ to find a physical quantity.

2 Adding Resistances

Problem [Cylindrical Resistor]

The cylindrical resistor shown in figure 1 is made such that the resistivity ρ is a function of the distance r from the axis. What is the total resistance R of the resistor?

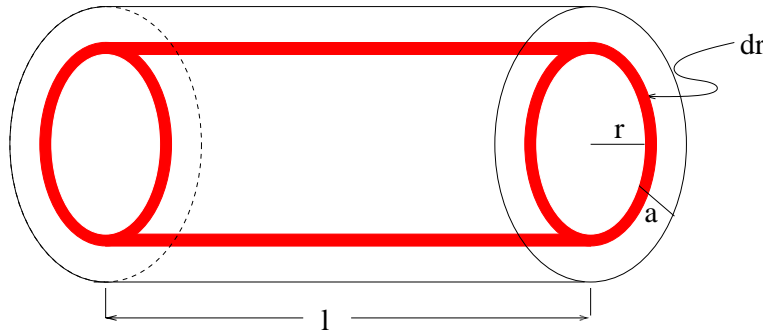


Figure 1: The figure shows a cylindrical wire of radius a . A potential difference is applied between the bases of the cylinder and therefore electric current is running parallel to the axis of the cylinder.

Towards a Solution:

We divide the cylindrical resistor into infinitesimal resistors in the form of cylindrical shells of thickness dr . One of these shells is seen in red in figure 1. If we apply equation (1) naively, we must write

$$R = \int_{\text{cylinder}} dR .$$

The infinitesimal resistance of the red shell is given by

$$dR = \rho(r) \frac{\ell}{dA} ,$$

where $dA = 2\pi r dr$ is the area of the base of the infinitesimal shell. Since dr is small, dR is huge, which is absurd. Where is the error?

Discussion:

When the current is flowing along the axis of the cylinder, the infinitesimal resistors are not connected in series. Therefore, the naive approach

$$R = \int dR$$

does not work since this formula assumes that the shells are connected in series. Instead, all of the infinitesimal cylindrical shells of width dr are connected at the same end points and, therefore, have the same applied potential. In other words, the shells are connected in parallel and it is the inverse resistance that is important, not R itself. Specifically

$$\frac{1}{R} = \int_{\text{cylinder}} d\left(\frac{1}{R}\right) .$$

Students may feel uncomfortable with this equation as at the beginning since it may seem ‘contradictory’ to their calculus knowledge; therefore, some discussion may be helpful.

Equation (1) states that when resistors are connected in series, they make it harder for the current to go through. Their resistances add to give the total resistance. However, when resistors are connected in parallel, many ‘paths’ are available simultaneously; the current is flowing easily and ‘resistance’ —which is a measure of flow difficulty— is not a good quantity to use. Maybe an analogy from everyday life is useful here. Paying tolls at toll booths is in direct analogy. When only a single booth is available, then all traffic has to go through that lane and no matter how dense the traffic is, there will be a relative delay. The traffic encounters some ‘resistance’ in the flow. However, when multiple booths are open, the drivers choose to go through the lanes that are free at the time of their approach to the booths and thus the delays encountered are minimal. In this case, the ‘availability’ of booths is a better quantity to be used to describe what is happening instead of the ‘resistance’ at the booths. Ultimately, the two quantities are related, but intuitively it is more satisfying to use one over the other depending on the situation. In direct analogy, for resistors connected in parallel, the relevant quantity is not R any longer, but S , where

$$S = \frac{1}{R} = \sigma \frac{A}{\ell} ,$$

and $\sigma = 1/\rho$ is the conductivity. We may call S the **conduction** of the resistor. When resistors are connected in parallel, they make it easier for the current to go through. Their conductions add to give the total conduction:

$$S = S_1 + S_2 + \cdots .$$

Thus, the conduction follows the usual addition

$$S = \int dS$$

when infinitesimal resistors are connected in parallel.

We are now in position to compute the answer to the posed problem in a way that is consistent with the intuition of the students.

Solution:

Since the cylindrical shells are connected in parallel, conduction is the additive quantity. For the infinitesimal shell

$$dS = \sigma(r) \frac{2\pi r dr}{\ell} .$$

Therefore

$$S = \int_{\text{cylinder}} dS = \frac{2\pi}{\ell} \int_0^a \sigma(r) r dr .$$

For example, if $\sigma(r) = \sigma_0 \frac{a}{r}$, then

$$S = 2\sigma_0 \frac{\pi a^2}{\ell} ,$$

where $\sigma_0 = 1/\rho_0$. The resistance is therefore

$$R = \frac{\rho_0}{2} \frac{\ell}{\pi a^2} .$$

Problem [Truncated-Cone Resistor]

A resistor is made from a truncated cone of material with uniform resistivity ρ . What is the total resistance R of the resistor when the potential difference is applied between the two bases of the cone?

Solution:

This is a well-known problem found in many of the introductory physics textbooks [2, 5, 6, 7, 8]. We can partition the cone into infinitesimal cylindrical resistors of length dz . One representative resistor at distance z from the top base is seen in figure 2. The area of the resistor is $A = \pi r^2$ and therefore its infinitesimal resistance is given by

$$dR = \rho \frac{dz}{\pi r^2} .$$

From the figure we can see that

$$\frac{z}{h} = \frac{r - b}{c - b} \Rightarrow dz = \frac{h}{c - b} dr .$$

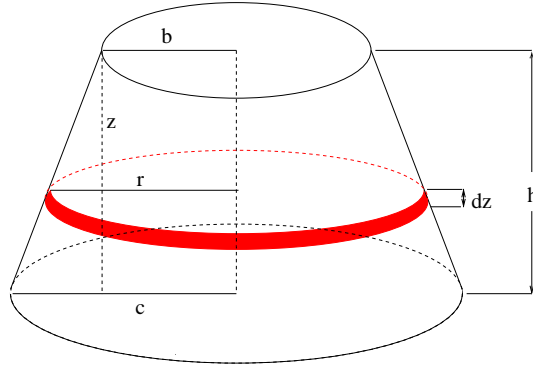


Figure 2: A truncated cone which has been sliced in infinitesimal cylinders of height dz .

The infinitesimal resistors are connected in series and therefore

$$R = \int_{\text{cone}} dR = \rho \frac{h}{\pi(c - b)} \int_b^c \frac{dr}{r^2} = \rho \frac{h}{\pi bc} . \quad (5)$$

3 Adding Capacitances

A similar discussion may be given for capacitors. When capacitors are connected in parallel, capacitance is the relative additive quantity:

$$C = C_1 + C_2 + \cdots .$$

For a parallel-plate capacitor of area A and distance d between the plates

$$C = \varepsilon_0 \frac{A}{d} .$$

When the capacitor is filled with a uniform dielectric of dielectric constant κ then

$$C = \varepsilon_0 \kappa \frac{A}{d} .$$

However, when capacitors are connected in series, the inverse capacitance

$$D = \frac{1}{C} .$$

is the additive quantity. We may call it the **incapacitance**. For a parallel-plate capacitor

$$D = \frac{1}{\varepsilon_0 \kappa} \frac{d}{A} .$$

In other words, when capacitors are connected in series

$$D = D_1 + D_2 + \cdots .$$

Problems like this are encountered when we fill a capacitor with a dielectric for which the dielectric constant is a function of the distance from the plates of the capacitor. Students are familiar with such problems for a parallel-plate capacitor in the discrete case. For example, problems asking students to compute the total capacitance in cases as those shown in figure 3 are found in several textbooks [5, 6, 8]. However, continuous problems are not found in any textbook [2, 3, 4, 5, 6, 7, 8].

We can easily construct new problems or re-work old problem using this idea. For example, the well-known formula for the capacitance of a cylindrical capacitor can be found this way. As shown in the left side of figure 4, the capacitor is partitioned into small cylindrical capacitors for which the distance between the plates is dr . For such small capacitors, the formula of a parallel-plate capacitor is valid. We notice though that all infinitesimal capacitors are connected in series. Therefore

$$dD = \frac{1}{\varepsilon_0} \frac{dr}{2\pi r h} .$$

and

$$D = \int_{\text{cylinder}} dD = \frac{1}{2\pi\varepsilon_0 h} \int_a^b \frac{dr}{r} = \frac{1}{2\pi\varepsilon_0 h} \ln \frac{a}{b} .$$

The total capacitance is then

$$C = \frac{1}{D} = \frac{2\pi\varepsilon_0 h}{\ln \frac{b}{a}} .$$

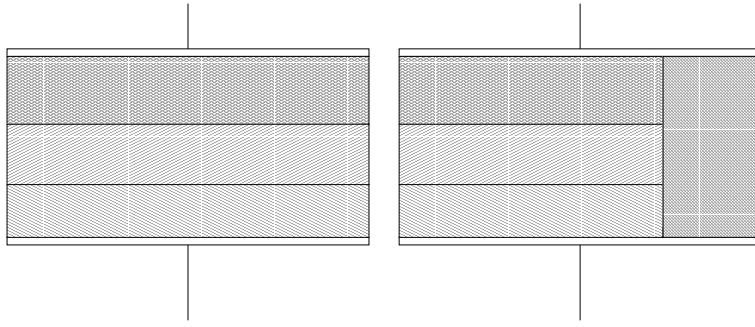


Figure 3: Two parallel-plate capacitors which are filled with uniform dielectrics of different dielectric constants.

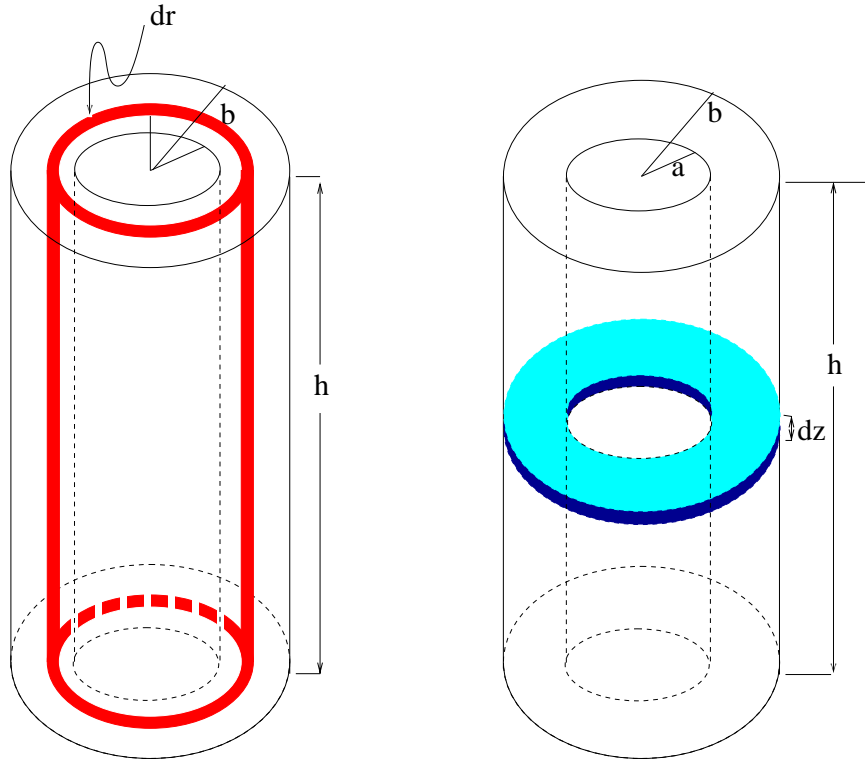


Figure 4: A cylindrical capacitor with radii a and b and height h . In the left picture, we have sliced it in infinitesimal cylindrical shells, while in the right picture we have sliced it in infinitesimal annuli.

Comment: One might be tempted to partition the cylindrical capacitor into infinitesimal capacitors as seen in the figure to the left (blue section). Such capacitors look simpler than the infinitesimal cylindrical shell we used above. Furthermore, they are connected in parallel (notice that each capacitor is carrying an infinitesimal charge dQ and $\int_{\text{cylinder}} dQ = Q$) and therefore it is enough to deal with capacitance, $C = \int_{\text{cylinder}} dC$, and not inductance D .

However, with a minute's reflection the reader will see that in order to use the parallel-plate capacitor formula in the infinitesimal case, the distance between the plates must be infinitesimal which indicates that the infinitesimal capacitors must be connected in series. In the proposed (blue) slicing, the distance between

the plates of the infinitesimal capacitor is finite, namely $b - a$. The infinitesimal capacitor is still a cylindrical capacitor of infinitesimal height and therefore its capacitance should be expressed in a form that is not known before the problem is solved. In other words,

$$dC = \frac{2\pi\epsilon_0}{\ln(b/a)} dz \quad (6)$$

from which

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)} \int_0^h dz = \frac{2\pi\epsilon_0}{\ln(b/a)} h . \quad (7)$$

But the expression (6) is unknown until the result (7) is found.

Problem [Truncated-cone Capacitor]

A capacitor is made of two circular disks of radii b and c respectively placed at a distance h such that the line that joins their centers is perpendicular to the disks. Find the capacitance of this arrangement (Seen in figure 2).

Solution: We partition the capacitor into infinitesimal parallel-plate capacitors of distance dz and plate area $A = \pi r^2$ exactly as seen in figure 2. These infinitesimal capacitors are connected in series and therefore the capacitance is the relevant additive quantity:

$$dD = \frac{1}{\epsilon_0} \frac{dz}{\pi r^2} .$$

Notice that the computation is identical to that of R with final result:

$$D = \frac{1}{\epsilon_0} \frac{h}{\pi bc} \Rightarrow C = \epsilon_0 \frac{\pi bc}{h} . \quad (8)$$

When $b = c$, we recover the result of the parallel-plate capacitor.

4 Conclusions

In this paper, we have tried to argue that when the right variables are used then the law of addition for capacitances and resistances is always additive. Table 1 summarizes our main formulæ. This is in agreement with the intuition of students when they solve continuous problems on the subject who like to add quantities in a simple way.

We must point out that our discussion by no means is restricted to capacitances and resistances only. Similar addition laws are encountered in many other areas of physics. For example, when connecting springs in parallel the total stiffness constant is given by the sum of the individual stiffness constants:

$$k = k_1 + k_2 + \cdots .$$

When the springs are connected in series, then

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots ,$$

pointing out that in this case, not the stiffness constant but the **elasticity constant**

$$\ell = \frac{1}{k}$$

is the relevant additive constant. Our discussion can thus be repeated verbatim in all similar cases. Table 2 lists the most common cases found in introductory physics.

	resistors	capacitors
series	$R = \sum_i R_i$	$D = \sum_i D_i$
parallel	$S = \sum_i S_i$	$C = \sum_i C_i$

Table 1: When viewed in the right physical quantities, addition of resistors and capacitors is always simple. In the above table, R is the resistance, S is the conduction, C is the capacitance, and D the incapacitance of a circuit element.

	resistors	capacitors	inductors	springs	thermal conductors
series	resistance R	incapacitance D	inductance L	elasticity ℓ	thermal resistance \mathcal{R}
parallel	conduction S	capacitance C	deductance K	stiffness k	thermal conduction \mathcal{S}

Table 2: This table summarizes the additive physical quantities in the most common cases encountered in introductory physics. The quantities that are not usually defined in the introductory books are the conduction $S = 1/R$, the incapacitance $D = 1/C$, the deductance $K = 1/L$, the elasticity constant $\ell = 1/k$, and the the thermal conduction $\mathcal{S} = 1/\mathcal{R}$.

A Suggested Problems

We end our article with some suggested problems which the reader may wish to solve.

1. Re-derive the well-known expression for the capacitance of a spherical capacitor

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} ,$$

(where a, b are the radii of the spheres with $b > a$) by partitioning it into infinitesimal capacitors.

2. Show that the capacitance of a cylindrical capacitor which is filled with a dielectric having dielectric constant $\kappa(r) = cr^n$, where r is the distance from the axis and $c, n \neq 0$ are constants, is given by

$$C = 2\pi\epsilon_0 hcn \frac{a^n b^n}{b^n - a^n} .$$

3. Show that the capacitance of a cylindrical capacitor which is filled with a dielectric having dielectric constant $\kappa(z) = cz^n$, where z is the distance from the basis and $c, n \geq 0$ are constants is given by

$$C = 2\pi\epsilon_0 \frac{ch^{n+1}}{(n+1) \ln(b/a)} .$$

4. Show that the capacitance of a spherical capacitor which is filled with a dielectric having dielectric constant $\kappa(r) = cr^n$, where r is the distance from the center and c, n are constants is given by

$$C = 4\pi\epsilon_0 \frac{c}{\ln(b/a)}$$

for $n = -1$ and

$$C = 4\pi\epsilon_0 c(n+1) \frac{a^{n+1}b^{n+1}}{b^{n+1} - a^{n+1}}$$

for $n \neq -1$.

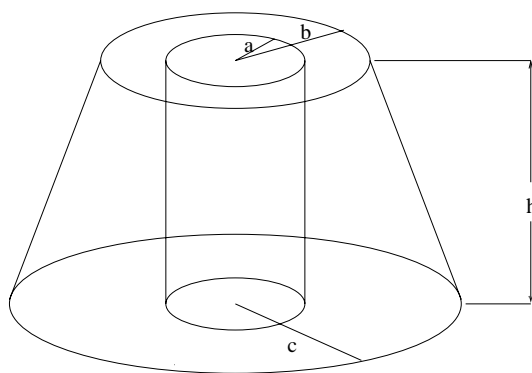


Figure 5: A hollow truncated cone

5. (a) Two metallic flat annuli are placed such that they form a capacitor with the shape of a hollow truncated cone as seen in figure 5. Partition the capacitor in infinitesimal capacitors and show that the capacitance is given by

$$C = 2\pi\epsilon_0 \frac{h}{a(c-b)} \left[\ln \frac{c-a}{c+a} - \ln \frac{b-a}{b+a} \right] .$$

Show that this result reduces to that of a cylindrical capacitor for $c = b$. Also, show that it agrees with (8) when $a = 0$.

(b) Now, fill the two bases with disks of radius a and argue that the capacitance of the hollow truncated cone equals that of the truncated cone minus the capacitance of the parallel-plate capacitor that we have removed (superposition principle). This means that the capacitance of the hollow truncated cone should equal to

$$C = \pi\epsilon_0 \frac{bc - a^2}{h} .$$

How is it possible that this result does not agree with that of part (a)?

6. A capacitor with the shape of a hollow truncated cone is now formed from two ‘cylindrical’ shells. Show that the capacitance in this case is

$$C = 2\pi\epsilon_0 \frac{ah}{c-b} \left[\text{li} \left(\frac{c}{a} \right) - \text{li} \left(\frac{b}{a} \right) \right] ,$$

where li is the logarithmic function[1]

$$\text{li}(X) \equiv \int_0^X \frac{dx}{\ln x} , \quad X > 1 .$$

7. (a) A conductor has the shape seen in figure 5. Show that the resistance when the voltage is applied between the upper and lower bases is given by

$$R = \rho \frac{h}{2\pi a(c-b)} \left[\ln \frac{c-a}{c+a} - \ln \frac{b-a}{b+a} \right] .$$

Show that this result reduces to equation (5) for $a = 0$.

(b) Argue now that the resistance of the hollow truncated-conical wire is the difference between the the resistance of the truncated-conical wire and a cylindrical wire of radius a (superposition principle). This implies that

$$R = \rho \frac{h}{bc - a^2} .$$

Explain why this does not agree with part (a).

8. A conductor has the shape of a hollow cylinder as seen in figure 4. Show that the resistance when the voltage is applied between the inner and outer surfaces is given by

$$R = \frac{\rho}{2\pi h} \ln \frac{b}{a} .$$

9. A conductor has the shape seen in figure 5. Show that the resistance when the voltage is applied between the inner and outer surfaces is given by

$$R = \frac{\rho}{2\pi h a} \frac{c - b}{\text{li}(c/a) - \text{li}(b/a)} .$$

Show that, for $c = b$, this result agrees with that of the previous problem.

10. A conductor has the shape of a truncated wedge as seen in figure 6. Show that the resistance of the conductor when the voltage is applied between the left and right faces is

$$R = \frac{\rho}{a} \frac{\ell \ln(c/b)}{c - b} ,$$

while the resistance when the voltage is applied between the top and bottom faces is

$$R = \frac{\rho}{a} \frac{c - b}{\ell \ln(c/b)} .$$

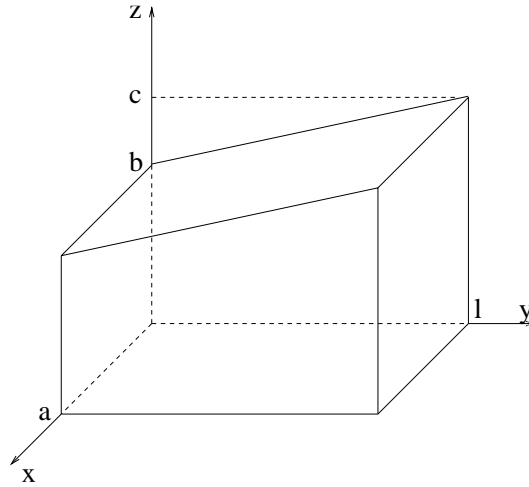


Figure 6: A conductor with the shape of a truncated wedge.

References

- [1] M. ABRAMOWITZ, I.A. STEGUN, Handbook of Mathematical Functions with Formulas, Dover.
- [2] D. HALIDAY, R. RESNICK, J. WALKER, Fundamentals of Physics, 6th ed., John-Wiley & Sons.
- [3] E. HECHT, Physics, Brooks/Cole 1996.
- [4] P. NOLAN, Fundamentals of College Physics, Wm. C. Brown Communications 1993.
- [5] R.A. SERWAY, Physics for Scientists and Engineers, 4th ed., Saunders College Publishing.
- [6] P.A. TIPLER, Physics for Scientists and Engineers, 3rd ed., Worth Publishers.
- [7] R. WOLFSON, J.M. PASACHOFF, Physics for Scientists and Engineers, 3rd ed., Addison-Wesley.
- [8] H.D. YOUNG, R.A. FREEDMAN, University Physics, 9th ed., Addison-Wesley 1996.